An Introduction to Operator Theory

1 Background

Definition 1.1. *Linear Spaces*

A *Linear space* is a set $L = \{x_1, x_2, \dots\}$ coupled with the following axioms:

1. We may perform "addition", $+$ on the elements of L, and for a finite number of these additions, the sum is still in L.
2. We may perform scalar multiplication on the elements of L, and the resulting object is still within L.
3. There exists a zero element with respect to addition. I.e., there exists $0 \in L$ such that $x + 0 = x$ for any $x \in L$.
4. For any $x, y \in L$, and scalar α
   (a) $x + y = y + x$.
   (b) $\alpha(x + y) = \alpha x + \alpha y$
   (c) $x - x = 0$

What happens if we want to talk about limits like $x_k \to x$ as $k \to \infty$? We need to define what "close" means.

Definition 1.2. *Metrics*

We call $D : A \times A \to \mathbb{R}$ a distance function, or *metric* on A if for any $x, y, z \in A$, D satisfies the following properties:

1. $D(x, y) = D(y, x)$
2. $D(x, x) = 0$
3. $D(x, y) \leq D(x, z) + D(z, y)$

Example 1.1.

Let L be the linear space of functions defined on $[0, 1]$, and let

$$D_1(f, g) = \max_{x \in [0,1]} \big|f(x) - g(x)\big|$$
$$D_2(f, g) = \int_0^1 \big|f(x) - g(x)\big|^2 \; dx$$

Definition 1.3. *Cauchy Sequences*

A sequence $\{x_n\}$ contained in a space, L, with metric, D, is *Cauchy iff* for any $\varepsilon > 0$, there exists $N \in N$ such that for all $m, n \geq N$, $D(x_n, x_m) < \varepsilon$. This can be formulated, in the sense of limits as:

$$\{x_n\} \text{ is Cauchy} \iff \lim_{\substack{m \to \infty \\ n \to \infty}} D(x_n, x_m) = 0$$

Exercise 1.1. *Prove that if $\{x_n\}$ is a sequence which converges to x, then x_n is Cauchy.*

Definition 1.4. *Complete Spaces*

A Linear space, L, is called *complete* if every cauchy sequence in L is convergent.

Definition 1.5. *Inner products*

We define the inner product of f and g (on L), as a complex number, denoted by (f, g), which satisfies the following properties for any $f, g \in L$ and α, β scalars:

1. $(f, g) = \overline{(g, f)}$
2. $(\alpha f_1 + \beta f_2, g) = \alpha(f_1, g) + \beta(f_2, g)$
3. $(f, f) \geq 0$ with $(f, f) = 0$ *iff* $f = 0$.

So if we have a Linear space, L, with no metric, and we have an inner product, $\langle f, g\rangle$, then we can define a metric on L by

$$D_L(f, g) = \sqrt{\langle f - g, f - g\rangle} \tag{1}$$

Exercise 1.2. *Show that D_L is a bonafied distance function.*

Definition 1.6. *Hilbert Spaces*

A Linear space with an inner product is called a *Hilbert Space* if it is complete with respect to the metric defined by (1).

Example 1.2. *Hilbert Space examples*

$L^2(a, b)$ — a function $f \in L^2(a, b)$ if $\int_a^b \left|f(x)\right|^2 \, dx < \infty$.

inner product: $\langle f, g\rangle_{L^2(a,b)} = \displaystyle\int_a^b f(x)\overline{g(x)}\, dx$

distance function: $D_{L^2(a,b)}(f, g) = \left\|f - g\right\|_{L^2(a,b)} = \int_a^b (f(x) - g(x))^2 \, dx$

Definition 1.7. *Open ball*

If $a \in A$, a Linear space with distance function D_A, we call $U(a, r)$ the *open ball* centered at a with radius r if

$$U(a, r) = \left\{x \in A \;\middle|\; D_A(x, a) < r\right\} \tag{2}$$

Definition 1.8. *Limit Points*

A point a in a space, L, is called a *limit point* of L if for any $\varepsilon > 0$, $U(a, \varepsilon) \setminus \{a\} \neq \emptyset$.

Definition 1.9. *Closure of a space*

If A is a space, and α_i are limit points of A, we call $\mathrm{cl}(A)$ the *closure* of A and define it as

$$\mathrm{cl}(A) = \bigcup_{i \in I} \alpha_i \tag{3}$$

$\mathrm{cl}(A)$ is sometimes denoted $\overline{A}$ (when this can't be confused with complex conjugation).

Definition 1.10. *A is closed...*

If A is a space, we say that A is *closed* if the closure of A is itself. I.e., A is closed *iff* $\overline{A} = A$.

Definition 1.11. *Denseness*

If S_1 and S_2 are two spaces, $(S_1 \subset S_2)$ then S_1 is *dense* in S_2 if $\overline{S_1} = S_2$. Alternatively, we could say that S_1 is dense in S_2 *iff* the existence of $f \in S_2$ such that $\langle f, g\rangle = 0$, for any $g \in S_1$ implied that $f \equiv 0$. That is, S_1 is dense in S_2 if the only perpendicular space to S_1 is $\{0\}$.

2 Operator Theory

Let H be a Hilbert Space with inner product $\langle f, g \rangle_H$, and let T be a linear operator acting in H[1]. What does it mean for T to be an operator?

Definition 2.1. *Operators*

$F : X \to Y$ is an *operator* if F assigns to each $x \in X$, a unique $y \in Y$. Essentially, operators are abstract functions.

2.1 Differential Operators

Example 2.1. *Suppose $H = L^2[0, 2\pi]$, and define*

$$L[y] = -y'' \qquad y = f(x), \qquad y(0) = y(2\pi)$$

The *Differential Operator* L is defined by the operation on y, $\ell[y]$, and boundary conditions.

Example 2.2. *Why boundary conditions are necessary to the definition of L*

Again, take $H = L^2[0, 2\pi]$, however let $\ell[y] = iy'$, and $y(0) = y(2\pi)$. Solving the Eigenvalue problem (i.e. $L[y] = \lambda y$), we find that

$$\begin{aligned} \lambda y &= iy' \\ \iff y' &= -i\lambda y \\ \iff y(t) &= \mathcal{C}e^{-i\lambda t} \end{aligned}$$

and using our boundary conditions:

$$\begin{aligned} y(0) &= \mathcal{C} \\ y(2\pi) &= \mathcal{C}e^{-i2\pi\lambda} \end{aligned}$$

So the eigenvalue problem boils down to solving when $e^{-i2\pi\lambda} = 1$, or equivalently, when $\cos(2\pi\lambda) = 1$, but this is whenever $\lambda \in \mathbb{Z}$. It should be clear that the problem of finding λ was directly influenced by our boundary conditions $y(0) = y(2\pi)$.

Definition 2.2. *Adjoints*

If T is a linear operator acting on a Hilbert space, H, with domain of definition $\text{dom}(T)$, then we call the mapping $g \to g^*$ such that

$$\langle T(f), g \rangle = \langle f, g^* \rangle \qquad (\forall\, f \in \text{dom}(T)) \tag{4}$$

the *adjoint* of T, and denote it T^*.

Exercise 2.1. *Prove T^* is an operator.*

Before we can continue we have to make the major assumption that $\overline{\text{dom}(T)} = H$.

Proof. It suffices to show that for a fixed $g \in H$, T^* cannot take g to g_1^* and g_2^* without $g_1^* = g_2^*$. So, $g \in H$ be given, f be any element in $\text{dom}(T)$, and suppose $g_1^*, g_2^* \in H$ satisfying (4). It follows that,

$$\begin{aligned} \langle T(f), g \rangle &= \langle f, g_1^* \rangle = \langle f, g_2^* \rangle \\ \iff \langle f, g_1^* - g_2^* \rangle &= 0 \end{aligned}$$

Now, this says that $g_1^* - g_2^* \in \text{dom}(T)^\perp$ (i.e. $g_1^* - g_2^*$ is orthogonal to every element in $\text{dom}(T)$). However, $\text{dom}(T)$ is dense in H, so the only space perpendicular to $\text{dom}(T)$ is the trivial space. Thus, $g_1^* - g_2^* = 0$. □

[1]Note: saying f is a function acting in a set X is equivalent to saying $f : X \to X$

Exercise 2.2. *Prove that* range$(T) \perp \ker(T^*)$.

Proof. If $x \in \text{range}(T)$ then it follows that there exists $y \in \text{dom}(T)$ such that $T(y) = x$. So, if $k \in \ker(T^*)$:

$$\begin{aligned}\langle x, k\rangle &= \langle T(y), k\rangle \\ &= \langle y, T^*(k)\rangle \\ &= \langle y, 0\rangle \\ &= 0\end{aligned}$$

Hence, $\text{range}(T) \perp \ker(T^*)$. ☐

Problem 2.1. *Given L, such that $\ell[y] = y'$, $y(0) = 0$, and $H = L^2[0, 2\pi]$, find L^*.*

Solution. From (4),

$$\begin{aligned}\langle L(f), g\rangle &= \int_0^{2\pi} f'(x)g(x)\, dx \\ &= f(x)g(x)\Big|_{x=0}^{x=2\pi} - \int_0^{2\pi} f(x)g'(x)\, dx \\ &= f(2\pi)g(2\pi) + \int_0^{2\pi} f(x)(-g'(x))\, dx\end{aligned}$$

Now if we impose the boundary condition that if $g \in \text{dom}(L^*)$ then $g(2\pi) = 0$, we see that

$$\langle L(f), g\rangle = \langle f, -L(g)\rangle$$

Hence, $L^*(g) = -L(g)$. So for $y \in \text{dom}(L^*)$, $\ell^*[y] = -y'$ and $y(2\pi) = 0$. ☐

Problem 2.2. *Define L as $\ell[y] = \cos(x)\, y''(x) - y'(x)$, $y(0) = y(1) = 0$, and $H = L^2[0, 1]$. Find L^*.*

Solution.

$$\begin{aligned}\langle L[f], g\rangle &= \int_0^1 (\cos(x)\, f''(x) - f'(x))\; g(x)\, dx \\ &= \underbrace{\int_0^1 \cos(x)\, f''(x)\, g(x)\, dx}_{I_1} - \underbrace{\int_0^1 f'(x)\, g(x)\, dx}_{I_2}\end{aligned}$$

Integrating I_1 by parts twice we get that

$$\begin{aligned}I_1 &= f'(x)\cos(x)\, g(x)\Big|_{x=0}^{x=1} + \int_0^1 f'(x)(\sin(x)g(x) + \cos(x)g'(x))\, dx \\ &= f'(x)\cos(x)\, g(x)\Big|_{x=0}^{x=1} + f(x)(\sin(x)\, g(x) + \cos(x)\, g'(x))\Big|_{x=0}^{x=1} - \int_0^1 f(x)\cos(x)(g(x) + g''(x))\, dx \\ &= f'(1)\cos(1)\, g(1) - f'(0)g(0) - \int_0^1 f(x)\cos(x)(g(x) + g''(x))\, dx\end{aligned}$$

and one round of integration by parts on I_2 tells us that

$$\begin{aligned}I_2 &= f(x)\, g(x)\Big|_{x=0}^{x=1} - \int_0^1 f(x)\, g'(x)\, dx \\ &= -\int_0^1 f(x)g'(x)\, dx\end{aligned}$$

Hence,

$$\begin{aligned}
\langle L[f], g\rangle &= I_1 - I_2 \\
&= f'(1)\cos(1)\, g(1) - f'(0)g(0) - \int_0^1 f(x)\cos(x)(g(x) + g''(x))\; dx + \int_0^1 f(x)g'(x)\; dx \\
&= f'(1)\cos(1)\, g(1) - f'(0)g(0) + \int_0^1 f(x)\Big(g'(x) - \cos(x)(g(x) + g''(x))\Big)\; dx
\end{aligned}$$

So if we impose the boundary conditions for $g \in \mathrm{dom}(L^*)$, $g(0) = g(1) = 0$, then we see that

$$\ell^*[g] = g'(x) - \cos(x)\big(g(x) + g''(x)\big)$$

□